Fractional Feynman-Kac equation for anomalous diffusion functionals

Shai Carmi Lior Turgeman Eli Barkai

Department of Physics
Bar-Ilan University

Weak Chaos, Infinite Ergodic Theory, and Anomalous Dynamics
Workshop, August 2011
Dresden, Germany

Section 1

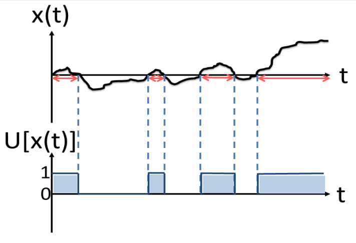
Introduction

Brownian functionals

- $x(t)$– the trajectory of a Brownian particle.
- The functional: $A \equiv \int_0^t U[x(\tau)]d\tau$.

Example

- $U(x) = \Theta(x) = \begin{cases} 1 & x > 0, \\ 0 & x < 0. \end{cases}$



- A– The occupation time in the positive half space.

The Feynman-Kac equation

- Denote $G(x, A, t)$ the joint PDF of x and A at time t.
- $G(x, p, t) = \int_0^\infty e^{-pA} G(x, A, t) dA$.
- The Feynman-Kac equation (M. Kac, 1949):

$$\boxed{\frac{\partial}{\partial t} G(x, p, t) = D \frac{\partial^2 G(x,p,t)}{\partial x^2} - pU(x) G(x, p, t)}$$

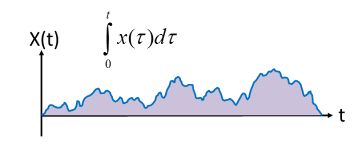
- D– the diffusion coefficient.
- $p = 0 \Rightarrow$ The diffusion equation $\frac{\partial}{\partial t} G(x, t) = D \frac{\partial^2 G(x,t)}{\partial x^2}$.
- Structurally similar to Schrödinger equation in imaginary time $i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + V(x) \Psi(x, t)$.

Examples of functionals and applications

$U(x) = x$

- The area under the random walk curve.
- The time-averaged position of the particle ($\overline{x} = \frac{1}{t}\int_0^t x(\tau)d\tau$).
- NMR (phase of each spin: $\varphi = \gamma\int_0^t B[x(\tau)]d\tau$, $B(x) = x$ is the magnetic field).

D.S. Grebenkov, *Rev. Mod. Phys.* **79**, 1077 (2007).



$U(x) = \begin{cases} 1 & \text{in a certain domain} \\ 0 & \text{otherwise} \end{cases}$

- The emitted light if fluorescent particles are excited only in the domain.

  G. Zumofen et al., *Phys. Rev. Lett.* **93**, 260601 (2004).
- The survival probability if the domain is absorbing.
- The distribution of the first passage time.
- The distribution of the maximal displacement.

A few more applications

$A = \int_0^t x(\tau)\Theta[x(\tau)]d\tau$

- The integrated excess temperature up to time t.
- S.N. Majumdar et al., *Phys. Rev. E* **6**5, 051112 (2002).

$A = \int_0^t x^2(\tau)d\tau$

- The fluctuations in the height of a KPZ surface.
- G. Foltin et al., *Phys. Rev. E* **5**0, R639 (1994).

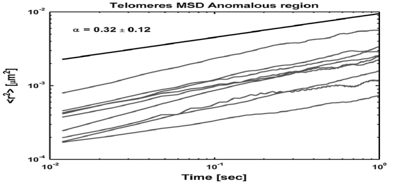
$A = \int_0^t e^{-\beta x(\tau)}d\tau$

- The partition function of a particle in a random potential (Sinai model).
- Models of stock prices.
- Reviewed in S.N. Majumdar, *Curr. Sci.* **89**, 2076 (2005).

Non-Brownian motion

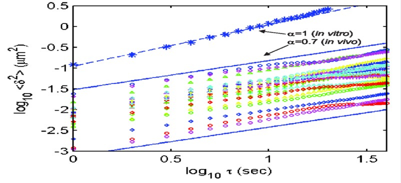
- In many physical and biological systems diffusion is anomalous.
- $\langle x^2 \rangle \sim t^{\alpha}$, $0 < \alpha < 1$.

DNA



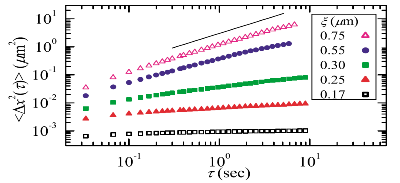
I. Bronstein et al., *Phys. Rev. Lett.* **103**, 018102 (2009).

RNA



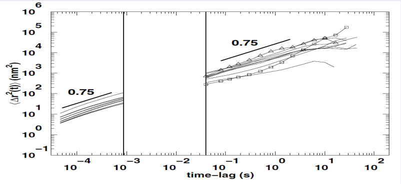
I. Golding et al., *Phys. Rev. Lett.* **96**, 098102 (2006).

Beads in Actin



I.Y. Wong et al., *Phys. Rev. Lett.* **92**, 178101 (2004).

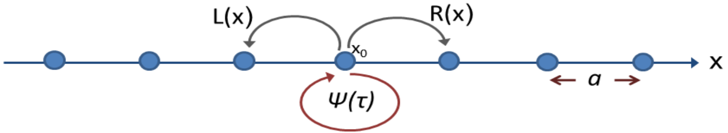
Lipid granules



I.M. Tolić-Nørrelykke et al., *Phys. Rev. Lett.* **93**, 078102 (2004).

Continuous-time random walk (CTRW)- A model for anomalous diffusion

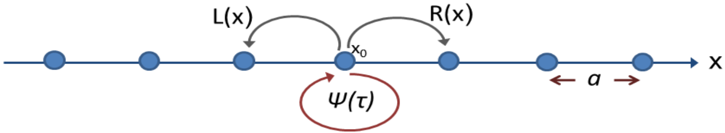
- An infinite 1D lattice with spacing a.
- Jumps to nearest neighbors.
  - For a free particle, $R(x) = L(x) = 1/2$.
  - For a bounded particle, $R(x)$ and $L(x)$ depend on the force at x.
- Start at x_0. Wait time τ and jump. Wait and jump again, ...
- Waiting times are distributed according to $\psi(\tau) \sim \tau^{-(1+\alpha)}$.



- $0 < \alpha < 1 \Rightarrow \langle \tau \rangle = \infty \Rightarrow \langle x^2 \rangle \sim t^{\alpha} \Rightarrow$ anomalous (sub-) diffusion.

Continuous-time random walk (CTRW)- A model for anomalous diffusion

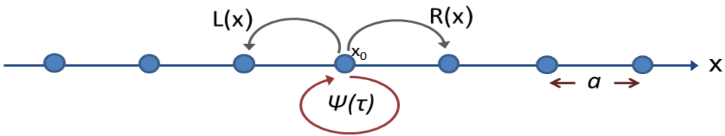
- An infinite 1D lattice with spacing a.
- Jumps to nearest neighbors.
  - For a free particle, $R(x) = L(x) = 1/2$.
  - For a bounded particle, $R(x)$ and $L(x)$ depend on the force at x.
- Start at x_0. Wait time τ and jump. Wait and jump again, ...
- Waiting times are distributed according to $\psi(\tau) \sim \tau^{-(1+\alpha)}$.



- $0 < \alpha < 1 \Rightarrow \langle \tau \rangle = \infty \Rightarrow \langle x^2 \rangle \sim t^\alpha \Rightarrow$ anomalous (sub-) diffusion.

Continuous-time random walk (CTRW)- A model for anomalous diffusion

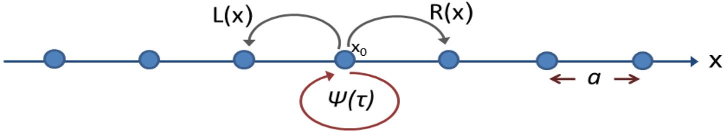
- An infinite 1D lattice with spacing a.
- Jumps to nearest neighbors.
  - For a free particle, $R(x) = L(x) = 1/2$.
  - For a bounded particle, $R(x)$ and $L(x)$ depend on the force at x.
- Start at x_0. Wait time τ and jump. Wait and jump again, ...
- Waiting times are distributed according to $\psi(\tau) \sim \tau^{-(1+\alpha)}$.



- $0 < \alpha < 1 \Rightarrow \langle \tau \rangle = \infty \Rightarrow \langle x^2 \rangle \sim t^{\alpha} \Rightarrow$ anomalous (sub-) diffusion.

Continuous-time random walk (CTRW)- A model for anomalous diffusion

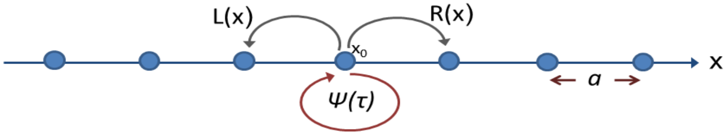
- An infinite 1D lattice with spacing a.
- Jumps to nearest neighbors.
  - For a free particle, $R(x) = L(x) = 1/2$.
  - For a bounded particle, $R(x)$ and $L(x)$ depend on the force at x.
- Start at x_0. Wait time τ and jump. Wait and jump again, ...
- Waiting times are distributed according to $\psi(\tau) \sim \tau^{-(1+\alpha)}$.



- $0 < \alpha < 1 \Rightarrow \langle \tau \rangle = \infty \Rightarrow \langle x^2 \rangle \sim t^\alpha \Rightarrow$ anomalous (sub-) diffusion.

Continuous-time random walk (CTRW)- A model for anomalous diffusion

- An infinite 1D lattice with spacing a.
- Jumps to nearest neighbors.
  - For a free particle, $R(x) = L(x) = 1/2$.
  - For a bounded particle, $R(x)$ and $L(x)$ depend on the force at x.
- Start at x_0. Wait time τ and jump. Wait and jump again, ...
- Waiting times are distributed according to $\psi(\tau) \sim \tau^{-(1+\alpha)}$.

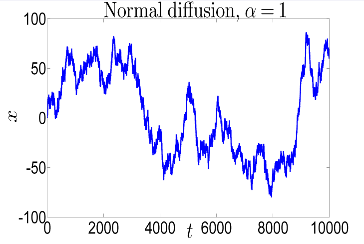


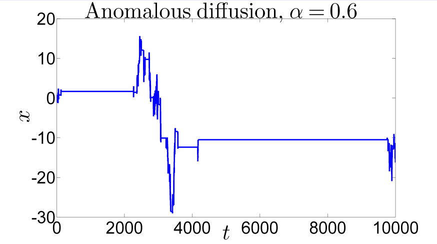
- $0 < \alpha < 1 \Rightarrow \langle \tau \rangle = \infty \Rightarrow \left\langle x^2 \right\rangle \sim t^\alpha \Rightarrow$ anomalous (sub-) diffusion.

CTRW example

CTRW is characterized by long periods of stagnation, whose lengths have no typical scale.

Sample trajectories





The fractional Fokker-Planck equation

- The continuum limit of CTRW is described by the fractional Fokker-Planck equation.
- Denote $G(x,t)$ the PDF of x at time t.

$$\boxed{\frac{\partial}{\partial t}G(x,t) = K_\alpha \mathcal{D}_t^{1-\alpha}\mathcal{L}_{\mathrm{FP}}G(x,t)}$$

- $\mathcal{L}_{\mathrm{FP}} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F(x)}{k_B T}$ (Fokker-Planck operator).
- $F(x) = 0 \Rightarrow$ the fractional diffusion equation.
- K_α— The generalized diffusion coefficient (units $\mathrm{m}^2/\mathrm{sec}^\alpha$).
- $\mathcal{D}_t^{1-\alpha}$ is a fractional derivative.
- What is a fractional derivative???

Fractional derivatives

- $\mathcal{D}_t^\alpha f(t) \equiv \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{f(\tau)}{(t-\tau)^\alpha}d\tau$ $(0 < \alpha < 1)$.
- In Laplace space: $\mathcal{D}_t^\alpha \tilde{f}(s) = s^\alpha \tilde{f}(s)$.
- Example: $\mathcal{D}_t^\alpha t^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha}$.
- Reduces to $\mathcal{D}_t^m t^n = \frac{n!}{(n-m)!}t^{n-m}$ for integer m, as expected.
- Composition of fractional derivatives makes sense (e.g., $\mathcal{D}_t^{1/2}\mathcal{D}_t^{1/2} = \frac{d}{dt}$).
- Riemann, Liouville, Cauchy.

The problem

- Many functionals are interesting in systems exhibiting anomalous diffusion.
- How can we find the PDF of these functionals?

In the following:

- Find an equation for the PDF of functionals of CTRW.
- Solve the equation for interesting functionals.
- Study implications to ergodicity breaking.

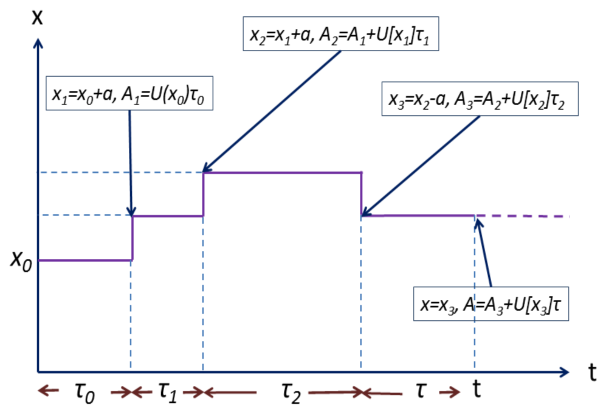
Section 2

The fractional Feynman-Kac equation

Illustration of the derivation method

- $A = \int_0^t U[x(\tau)d\tau = U(x_0)\tau_0 + U(x_1)\tau_1 + U(x_2)\tau_2 + ...$

A trajectory and its corresponding functional



Outline of the derivation

- $W(\tau) = 1 - \int_0^\tau \psi(\tau')d\tau'$:
  The probability not to jump until time τ.
- $\chi(x, A, t)dt$– the probability to jump into (x, A) during $[t, t + dt]$.
- $G(x, A, t) = \int_0^t W(\tau)\chi[x, A - \tau U(x), t - \tau]d\tau$.
- $$\chi(x, A, t) = \int_0^t \psi(\tau)L(x + a)\chi[x + a, A - \tau U(x + a), t - \tau]d\tau + \int_0^t \psi(\tau)R(x - a)\chi[x - a, A - \tau U(x - a), t - \tau]d\tau + \delta(x - x_0)\delta(A)\delta(t).$$
- Laplace-Fourier transform $A \to p$, $t \to s$, $x \to k$.
- $$\chi(k, p, s) = e^{-ika} \int_{-\infty}^{\infty} e^{ikx}L(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx + e^{ika} \int_{-\infty}^{\infty} e^{ikx}R(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx + e^{ikx_0}.$$
- Use detailed balance and assume thermal equilibrium to show:
  $R(x) \approx \frac{1}{2}\left[1 + \frac{aF(x)}{2k_BT}\right] = 1 - L(x)$.

Outline of the derivation

- $W(\tau) = 1 - \int_0^{\tau} \psi(\tau')d\tau'$:
  The probability not to jump until time τ.
- $\chi(x, A, t)dt$– the probability to jump into (x, A) during $[t, t + dt]$.
- $G(x, A, t) = \int_0^t W(\tau)\chi[x, A - \tau U(x), t - \tau]d\tau$.
- $$\begin{aligned}\chi(x, A, t) = &\int_0^t \psi(\tau)L(x + a)\chi[x + a, A - \tau U(x + a), t - \tau]d\tau \\ &+ \int_0^t \psi(\tau)R(x - a)\chi[x - a, A - \tau U(x - a), t - \tau]d\tau \\ &+ \delta(x - x_0)\delta(A)\delta(t).\end{aligned}$$
- Laplace-Fourier transform $A \to p$, $t \to s$, $x \to k$.
- $$\begin{aligned}\chi(k, p, s) = &e^{-ika} \int_{-\infty}^{\infty} e^{ikx} L(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx \\ &+e^{ika} \int_{-\infty}^{\infty} e^{ikx} R(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx \\ &+e^{ikx_0}.\end{aligned}$$
- Use detailed balance and assume thermal equilibrium to show:
  $R(x) \approx \frac{1}{2}\left[1 + \frac{aF(x)}{2k_BT}\right] = 1 - L(x)$.

Outline of the derivation

- $W(\tau) = 1 - \int_0^\tau \psi(\tau')d\tau'$:
  The probability not to jump until time τ.
- $\chi(x, A, t)dt$– the probability to jump into (x, A) during $[t, t + dt]$.
- $G(x, A, t) = \int_0^t W(\tau)\chi[x, A - \tau U(x), t - \tau]d\tau$.
- $\chi(x, A, t) = \int_0^t \psi(\tau)L(x + a)\chi[x + a, A - \tau U(x + a), t - \tau]d\tau$
  $+ \int_0^t \psi(\tau)R(x - a)\chi[x - a, A - \tau U(x - a), t - \tau]d\tau$
  $+ \delta(x - x_0)\delta(A)\delta(t)$.
- Laplace-Fourier transform $A \to p$, $t \to s$, $x \to k$.
- $\chi(k, p, s) = e^{-ika} \int_{-\infty}^{\infty} e^{ikx}L(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx$
  $+e^{ika} \int_{-\infty}^{\infty} e^{ikx}R(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx$
  $+e^{ikx_0}$.
- Use detailed balance and assume thermal equilibrium to show:
  $R(x) \approx \frac{1}{2}\left[1 + \frac{aF(x)}{2k_BT}\right] = 1 - L(x)$.

Outline of the derivation

- $W(\tau) = 1 - \int_0^\tau \psi(\tau')d\tau'$:
  The probability not to jump until time τ.
- $\chi(x, A, t)dt$– the probability to jump into (x, A) during $[t, t + dt]$.
- $G(x, A, t) = \int_0^t W(\tau)\chi[x, A - \tau U(x), t - \tau]d\tau$.
- $\chi(x, A, t) = \int_0^t \psi(\tau)L(x + a)\chi[x + a, A - \tau U(x + a), t - \tau]d\tau$
  $+ \int_0^t \psi(\tau)R(x - a)\chi[x - a, A - \tau U(x - a), t - \tau]d\tau$
  $+ \delta(x - x_0)\delta(A)\delta(t)$.
- Laplace-Fourier transform $A \to p$, $t \to s$, $x \to k$.
- $\chi(k, p, s) = e^{-ika} \int_{-\infty}^{\infty} e^{ikx} L(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx$
  $+e^{ika} \int_{-\infty}^{\infty} e^{ikx} R(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx$
  $+e^{ikx_0}$.
- Use detailed balance and assume thermal equilibrium to show:
  $R(x) \approx \frac{1}{2}\left[1 + \frac{aF(x)}{2k_BT}\right] = 1 - L(x)$.

Outline of the derivation

- $W(\tau) = 1 - \int_0^\tau \psi(\tau')d\tau'$:
  The probability not to jump until time τ.
- $\chi(x, A, t)dt$– the probability to jump into (x, A) during $[t, t + dt]$.
- $G(x, A, t) = \int_0^t W(\tau)\chi[x, A - \tau U(x), t - \tau]d\tau$.
- $$\begin{aligned}\chi(x, A, t) = & \int_0^t \psi(\tau)L(x + a)\chi[x + a, A - \tau U(x + a), t - \tau]d\tau \\ & + \int_0^t \psi(\tau)R(x - a)\chi[x - a, A - \tau U(x - a), t - \tau]d\tau \\ & + \delta(x - x_0)\delta(A)\delta(t).\end{aligned}$$
- Laplace-Fourier transform $A \to p$, $t \to s$, $x \to k$.
- $$\begin{aligned}\chi(k, p, s) = & e^{-ika} \int_{-\infty}^{\infty} e^{ikx}L(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx \\ & + e^{ika} \int_{-\infty}^{\infty} e^{ikx}R(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx \\ & + e^{ikx_0}.\end{aligned}$$
- Use detailed balance and assume thermal equilibrium to show:
  $R(x) \approx \frac{1}{2}\left[1 + \frac{aF(x)}{2k_BT}\right] = 1 - L(x)$.

Outline of the derivation

- $W(\tau) = 1 - \int_0^\tau \psi(\tau')d\tau'$:
  The probability not to jump until time τ.
- $\chi(x, A, t)dt$– the probability to jump into (x, A) during $[t, t + dt]$.
- $G(x, A, t) = \int_0^t W(\tau)\chi[x, A - \tau U(x), t - \tau]d\tau$.
- $$\begin{aligned}\chi(x, A, t) = &\int_0^t \psi(\tau)L(x + a)\chi[x + a, A - \tau U(x + a), t - \tau]d\tau \\ &+ \int_0^t \psi(\tau)R(x - a)\chi[x - a, A - \tau U(x - a), t - \tau]d\tau \\ &+ \delta(x - x_0)\delta(A)\delta(t).\end{aligned}$$
- Laplace-Fourier transform $A \to p$, $t \to s$, $x \to k$.
- $$\begin{aligned}\chi(k, p, s) = &e^{-ika} \int_{-\infty}^{\infty} e^{ikx}L(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx \\ &+e^{ika} \int_{-\infty}^{\infty} e^{ikx}R(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx \\ &+e^{ikx_0}.\end{aligned}$$
- Use detailed balance and assume thermal equilibrium to show:
  $R(x) \approx \frac{1}{2}\left[1 + \frac{aF(x)}{2k_BT}\right] = 1 - L(x)$.

Outline of the derivation

- $W(\tau) = 1 - \int_0^\tau \psi(\tau')d\tau'$:
  The probability not to jump until time τ.
- $\chi(x, A, t)dt$– the probability to jump into (x, A) during $[t, t + dt]$.
- $G(x, A, t) = \int_0^t W(\tau)\chi[x, A - \tau U(x), t - \tau]d\tau$.
- $$\begin{aligned}\chi(x, A, t) = &\int_0^t \psi(\tau)L(x + a)\chi[x + a, A - \tau U(x + a), t - \tau]d\tau \\ &+ \int_0^t \psi(\tau)R(x - a)\chi[x - a, A - \tau U(x - a), t - \tau]d\tau \\ &+ \delta(x - x_0)\delta(A)\delta(t).\end{aligned}$$
- Laplace-Fourier transform $A \to p$, $t \to s$, $x \to k$.
- $$\begin{aligned}\chi(k, p, s) = &e^{-ika} \int_{-\infty}^{\infty} e^{ikx}L(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx \\ &+e^{ika} \int_{-\infty}^{\infty} e^{ikx}R(x)\hat{\psi}[s + pU(x)]\chi(x, p, s)dx \\ &+e^{ikx_0}.\end{aligned}$$
- Use detailed balance and assume thermal equilibrium to show:
  $R(x) \approx \frac{1}{2}\left[1 + \frac{aF(x)}{2k_BT}\right] = 1 - L(x)$.

Outline of the derivation

- $G(k,p,s) = \frac{1-\hat{\psi}\left[s+pU\left(-i\frac{\partial}{\partial k}\right)\right]}{s+pU\left(-i\frac{\partial}{\partial k}\right)} \times$
  $\times \left\{1 - \left[\cos(ka) + i\sin(ka)\frac{aF\left(-i\frac{\partial}{\partial k}\right)}{2k_BT}\right]\hat{\psi}\left[s+pU\left(-i\frac{\partial}{\partial k}\right)\right]\right\}^{-1} e^{ikx_0}.$
- Apply the continuum approximation $\hat{\psi}(s) \sim 1 - B_\alpha s^\alpha$, $\cos(ka) \sim 1 - \frac{1}{2}a^2k^2$, and $K_\alpha = a^2/(2B_\alpha)$.
- $sG(k,p,s) - e^{ikx_0} =$
  $-K_\alpha \left[k^2 - ik\frac{F\left(-i\frac{\partial}{\partial k}\right)}{k_BT}\right]\left[s+pU\left(-i\frac{\partial}{\partial k}\right)\right]^{1-\alpha} G(k,p,s) -$
  $pU\left(-i\frac{\partial}{\partial k}\right)G(k,p,s).$

Outline of the derivation

- $G(k,p,s) = \frac{1-\hat{\psi}\left[s+pU\left(-i\frac{\partial}{\partial k}\right)\right]}{s+pU\left(-i\frac{\partial}{\partial k}\right)}\times$

  $\times\left\{1-\left[\cos(ka)+i\sin(ka)\frac{aF\left(-i\frac{\partial}{\partial k}\right)}{2k_BT}\right]\hat{\psi}\left[s+pU\left(-i\frac{\partial}{\partial k}\right)\right]\right\}^{-1}e^{ikx_0}.$
- Apply the continuum approximation $\hat{\psi}(s)\sim 1-B_\alpha s^\alpha$, $\cos(ka)\sim 1-\frac{1}{2}a^2k^2$, and $K_\alpha = a^2/(2B_\alpha)$.
- $sG(k,p,s)-e^{ikx_0} =$
  $-K_\alpha\left[k^2-ik\frac{F\left(-i\frac{\partial}{\partial k}\right)}{k_BT}\right]\left[s+pU\left(-i\frac{\partial}{\partial k}\right)\right]^{1-\alpha}G(k,p,s)-$
  $pU\left(-i\frac{\partial}{\partial k}\right)G(k,p,s).$

The fractional Feynman-Kac equation

- The fractional Feynman-Kac equation:

$$\boxed{\frac{\partial}{\partial t}G(x,p,t) = K_\alpha \mathcal{L}_{\rm FP}\mathcal{D}_t^{1-\alpha}G(x,p,t) - pU(x)G(x,p,t)}$$

- $\mathcal{L}_{\rm FP} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F(x)}{k_B T}$.
- Here, $\mathcal{D}_t^{1-\alpha}G(x,p,t) = \frac{1}{\Gamma(\alpha)}\left[\frac{\partial}{\partial t} + pU(x)\right]\int_0^t \frac{e^{-(t-\tau)pU(x)}}{(t-\tau)^{1-\alpha}}G(x,p,\tau)d\tau$.
- $\mathcal{D}_t^{1-\alpha}G(x,p,s) = [s+pU(x)]^{1-\alpha}G(x,p,s)$.
- Substantial fractional derivative operator (Friedrich/Metzler).

- $\alpha = 1 \Rightarrow$ The Feynman-Kac equation.
- $p = 0 \Rightarrow$ The fractional Fokker-Planck equation.
- For a free particle, $\mathcal{L}_{\rm FP} = \frac{\partial^2}{\partial x^2}$.

The fractional Feynman-Kac equation

- The fractional Feynman-Kac equation:

$$\boxed{\frac{\partial}{\partial t}G(x,p,t) = K_\alpha \mathcal{L}_{\mathrm{FP}} \mathcal{D}_t^{1-\alpha} G(x,p,t) - pU(x)G(x,p,t)}$$

- $\mathcal{L}_{\mathrm{FP}} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F(x)}{k_B T}$.
- Here, $\mathcal{D}_t^{1-\alpha} G(x,p,t) = \frac{1}{\Gamma(\alpha)}\left[\frac{\partial}{\partial t} + pU(x)\right]\int_0^t \frac{e^{-(t-\tau)pU(x)}}{(t-\tau)^{1-\alpha}} G(x,p,\tau)d\tau$.
- $\mathcal{D}_t^{1-\alpha} G(x,p,s) = [s + pU(x)]^{1-\alpha} G(x,p,s)$.
- Substantial fractional derivative operator (Friedrich/Metzler).

- $\alpha = 1 \;\Rightarrow$ The Feynman-Kac equation.
- $p = 0 \;\Rightarrow$ The fractional Fokker-Planck equation.
- For a free particle, $\mathcal{L}_{\mathrm{FP}} = \frac{\partial^2}{\partial x^2}$.

Extensions

General $U(x)$ (not necessarily positive)

- *Fourier* transform $A \to p$.
- $\frac{\partial}{\partial t} G(x,p,t) = K_\alpha \frac{\partial^2}{\partial x^2} \mathcal{D}_t^{1-\alpha} G(x,p,t) + ipU(x)G(x,p,t)$.

Lévy flights

- Displacements Δ_x have PDF $f(\Delta_x) \sim |\Delta_x|^{-(1+\mu)}$ $(0 < \mu < 2)$.
- $\frac{\partial}{\partial t} G(x,p,t) = K_{\alpha,\mu} \nabla_x^\mu \mathcal{D}_t^{1-\alpha} G(x,p,t) - pU(x)G(x,p,t)$.
- $\nabla_x^\mu \to -|k|^\mu$ is the Riesz spatial fractional derivative operator.

Time dependent force

- $R(x)$ and $L(x)$ depend on the force at the time of the jump.
- $\frac{\partial}{\partial t} G(x,p,t) = K_\alpha \mathcal{L}_{\mathrm{FP}} \mathcal{D}_t^{1-\alpha} G(x,p,t) - pU(x)G(x,p,t)$.
- $\mathcal{L}_{\mathrm{FP}} = \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \frac{F(x,t)}{k_B T}$.

How to solve the equation

A recipe

1. Solve the fractional Feynman-Kac equation in (p, s) space.
$sG(x, p, s) - \delta(x - x_0) =$
$K_\alpha \mathcal{L}_{\mathrm{FP}}[s + pU(x)]^{1-\alpha} G(x, p, s) - pU(x)G(x, p, s).$
2. Integrate the solution over all x to eliminate the dependence on the final position of the particle.
3. Invert $(p, s) \to (A, t)$, to obtain $G(A, t)$, the PDF of A at time t.

The backward equation

- We would like to avoid the x-dependence.
- $$G_{x_0}(A,t) = \int_0^t \psi(\tau)R(x_0)G_{x_0+a}[A-\tau U(x_0), t-\tau]d\tau$$
  $$+\int_0^t \psi(\tau)L(x_0)G_{x_0-a}[A-\tau U(x_0), t-\tau]d\tau$$
  $$+W(t)\delta[A-tU(x_0)].$$
- $G_{x_0}(A,t)$— the PDF of A when the process started at x_0.
- The *backward* fractional Feynman-Kac equation:

$$\boxed{\frac{\partial}{\partial t}G_{x_0}(p,t) = K_\alpha \mathcal{D}_t^{1-\alpha}\mathcal{L}_{\text{FP}}^{(\text{B})}G_{x_0}(p,t) - pU(x_0)G_{x_0}(p,t)}$$

- $\mathcal{L}_{\text{FP}}^{(\text{B})} = \frac{\partial^2}{\partial x_0^2} + \frac{F(x_0)}{k_B T}\frac{\partial}{\partial x_0}$.

Section 3

Functionals of free particles

Functionals of free particles

Functional	Result
The PDF of the occupation fraction in $x>0$	$G(\lambda)=\frac{\sin(\pi\alpha/2)}{\pi}\frac{\lambda^{\alpha/2-1}(1-\lambda)^{\alpha/2-1}}{\lambda^{\alpha}+(1-\lambda)^{\alpha}+2\lambda^{\alpha/2}(1-\lambda)^{\alpha/2}\cos(\pi\alpha/2)}$
The first passage time from 0 to b	$f(t_f)=\frac{1}{\tau_f}l_{\alpha/2}\left(\frac{t_f}{\tau_f}\right);\ \tau_f=(b^2/K_\alpha)^{1/\alpha}$
The maximal displacement	$P(x_m,t)=\frac{2}{\alpha\sqrt{K_\alpha}}\frac{t}{\left(x_m/\sqrt{K_\alpha}\right)^{1+2/\alpha}}l_{\alpha/2}\left[\frac{t}{\left(x_m/\sqrt{K_\alpha}\right)^{2/\alpha}}\right]$
The probability of hitting L before 0	$Q_L(x_0)=\frac{x_0}{L}$
The first two moments of the time in $[-b,b]$	$\langle T_i\rangle\sim\frac{b}{\sqrt{K_\alpha}\Gamma(2-\alpha/2)}t^{1-\alpha/2}\quad;\quad\langle T_i^2\rangle\sim\frac{2b(1-\alpha)}{\sqrt{K_\alpha}\Gamma(3-\alpha/2)}t^{2-\alpha/2}$
The survival probability when $[-b,b]$ absorbing with rate R	$\langle S\rangle\sim\left[\Gamma(1-\alpha/2)\sinh\left(\frac{bR^{\alpha/2}}{\sqrt{K_\alpha}}\right)\right]^{-1}(Rt)^{-\alpha/2}$
The fluctuations of the time-averaged position	$\left\langle(\Delta\overline{x})^2\right\rangle=\frac{4K_\alpha}{\Gamma(3+\alpha)}t^\alpha$
A scaling form for the distribution of $U(x)=x^k$ functionals	$G(A_{x^k},t)=\frac{1}{K_\alpha^{k/2}t^{1+\alpha k/2}}g_{\alpha,k}\left(\frac{A_{x^k}}{K_\alpha^{k/2}t^{1+\alpha k/2}}\right)$

The occupation time in the positive half-space

- $T_+ \equiv \int_0^t \Theta[x(\tau)]d\tau$, or $U(x) = \Theta(x)$.
- The backward equation is:
$$sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) & x_0 < 0, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > 0. \end{cases}$$
- Solve in each half-space separately, and match G_{x_0} and its derivative. Substitute $x_0 = 0$.
- $G_0(p,s) = \frac{s^{\alpha/2-1} + (s+p)^{\alpha/2-1}}{s^{\alpha/2} + (s+p)^{\alpha/2}}$.
- The PDF of $\lambda \equiv T_+/t$ is the *Lamperti* PDF:
$G(\lambda) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{\lambda^{\alpha/2-1}(1-\lambda)^{\alpha/2-1}}{\lambda^\alpha + (1-\lambda)^\alpha + 2\lambda^{\alpha/2}(1-\lambda)^{\alpha/2}\cos(\pi\alpha/2)}$.

The occupation time in the positive half-space

- $T_+ \equiv \int_0^t \Theta[x(\tau)]d\tau$, or $U(x) = \Theta(x)$.
- The backward equation is:
$$sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) & x_0 < 0, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > 0. \end{cases}$$
- Solve in each half-space separately, and match G_{x_0} and its derivative. Substitute $x_0 = 0$.
- $G_0(p,s) = \frac{s^{\alpha/2-1} + (s+p)^{\alpha/2-1}}{s^{\alpha/2} + (s+p)^{\alpha/2}}$.
- The PDF of $\lambda \equiv T_+/t$ is the *Lamperti* PDF:
$$G(\lambda) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{\lambda^{\alpha/2-1}(1-\lambda)^{\alpha/2-1}}{\lambda^\alpha + (1-\lambda)^\alpha + 2\lambda^{\alpha/2}(1-\lambda)^{\alpha/2}\cos(\pi\alpha/2)}.$$

The occupation time in the positive half-space

- $T_+ \equiv \int_0^t \Theta[x(\tau)]d\tau$, or $U(x) = \Theta(x)$.
- The backward equation is:
$$sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) & x_0 < 0, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > 0. \end{cases}$$
- Solve in each half-space separately, and match G_{x_0} and its derivative. Substitute $x_0 = 0$.
- $G_0(p,s) = \frac{s^{\alpha/2-1} + (s+p)^{\alpha/2-1}}{s^{\alpha/2} + (s+p)^{\alpha/2}}$.
- The PDF of $\lambda \equiv T_+/t$ is the *Lamperti* PDF:
$$G(\lambda) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{\lambda^{\alpha/2-1}(1-\lambda)^{\alpha/2-1}}{\lambda^\alpha + (1-\lambda)^\alpha + 2\lambda^{\alpha/2}(1-\lambda)^{\alpha/2}\cos(\pi\alpha/2)}.$$

The occupation time in the positive half-space

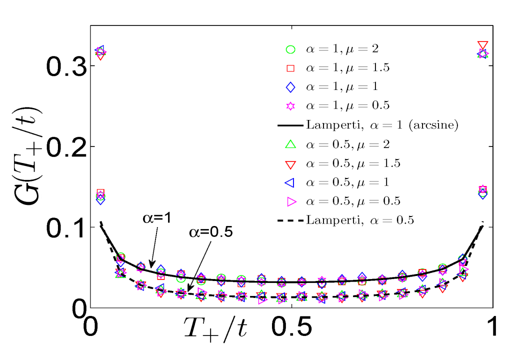
- $T_+ \equiv \int_0^t \Theta[x(\tau)]d\tau$, or $U(x) = \Theta(x)$.
- The backward equation is:
$$sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) & x_0 < 0, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > 0. \end{cases}$$
- Solve in each half-space separately, and match G_{x_0} and its derivative. Substitute $x_0 = 0$.
- $G_0(p,s) = \frac{s^{\alpha/2-1} + (s+p)^{\alpha/2-1}}{s^{\alpha/2} + (s+p)^{\alpha/2}}$.
- The PDF of $\lambda \equiv T_+/t$ is the *Lamperti* PDF:
$G(\lambda) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{\lambda^{\alpha/2-1}(1-\lambda)^{\alpha/2-1}}{\lambda^\alpha + (1-\lambda)^\alpha + 2\lambda^{\alpha/2}(1-\lambda)^{\alpha/2}\cos(\pi\alpha/2)}$.

The occupation time in the positive half-space

- $T_+ \equiv \int_0^t \Theta[x(\tau)]d\tau$, or $U(x) = \Theta(x)$.
- The backward equation is:
$$sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) & x_0 < 0, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial {x_0}^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > 0. \end{cases}$$
- Solve in each half-space separately, and match G_{x_0} and its derivative. Substitute $x_0 = 0$.
- $G_0(p,s) = \frac{s^{\alpha/2-1} + (s+p)^{\alpha/2-1}}{s^{\alpha/2} + (s+p)^{\alpha/2}}$.
- The PDF of $\lambda \equiv T_+/t$ is the *Lamperti* PDF:
$$G(\lambda) = \frac{\sin(\pi\alpha/2)}{\pi} \frac{\lambda^{\alpha/2-1}(1-\lambda)^{\alpha/2-1}}{\lambda^\alpha + (1-\lambda)^\alpha + 2\lambda^{\alpha/2}(1-\lambda)^{\alpha/2}\cos(\pi\alpha/2)}.$$

The occupation time in the positive half-space

Lamperti's PDF



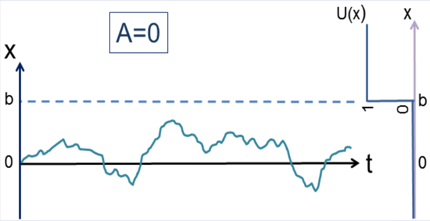
- The particle is almost all the time in one half!

- For $\alpha = 1$, reduces to the famous *Lévy's arcsine law*: $G(\lambda) = \frac{1}{\pi\sqrt{\lambda(1-\lambda)}}$.
- Occupation time PDF is Lamperti's even for Lévy flights.

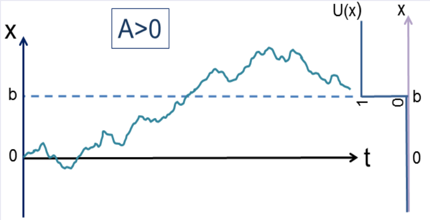
The first passage time

- The time t_f when a particle starting at $x_0 = 0$ first hits $x = b > 0$ is called the first passage time.
- A trick due to Kac, 1951:
- $\Pr\{t_f > t\} = \Pr\left\{\max_{0 \le \tau \le t} x(\tau) < b\right\} = \lim_{p \to \infty} G_{x_0}(p, t)$
- $U(x) = \begin{cases} 0 & x < b, \\ 1 & x > b. \end{cases}$
- Works because $G_{x_0}(p, t) = \int_0^\infty e^{-pA} G_{x_0}(A, t) dA$.

$t_f > t$



$t_f < t$



The first passage time

- $sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) & x_0 < b, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > b. \end{cases}$
- Solve, take the limit $p \to \infty$, and invert:
- $\lim_{p\to\infty} G_0(p,t) = \Pr\{t_f > t\} = 1 - \int_0^t \frac{1}{\tau_f} l_{\alpha/2}\left(\frac{\tau}{\tau_f}\right) d\tau.$
  - $\tau_f = (b^2/K_\alpha)^{1/\alpha}$.
  - $l_{\alpha/2}(t)$ is the one-sided Lévy distribution of order $\alpha/2$.
  - $\tilde{l}_{\alpha/2}(s) = e^{-s^{\alpha/2}}$.
- The first passage time PDF: $f(t_f) = \frac{1}{\tau_f} l_{\alpha/2}\left(\frac{t_f}{\tau_f}\right)$.
- For long times, $f(t_f) \sim t_f^{-(1+\alpha/2)}$ (famous $t_f^{-3/2}$ for $\alpha = 1$).

The first passage time

- $sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) & x_0 < b, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > b. \end{cases}$
- Solve, take the limit $p \to \infty$, and invert:
- $\lim_{p\to\infty} G_0(p,t) = \Pr\{t_f > t\} = 1 - \int_0^t \frac{1}{\tau_f} l_{\alpha/2}\left(\frac{\tau}{\tau_f}\right) d\tau.$
  - $\tau_f = (b^2/K_\alpha)^{1/\alpha}$.
  - $l_{\alpha/2}(t)$ is the one-sided Lévy distribution of order $\alpha/2$.
  - $\tilde{l}_{\alpha/2}(s) = e^{-s^{\alpha/2}}$.
- The first passage time PDF: $f(t_f) = \frac{1}{\tau_f} l_{\alpha/2}\left(\frac{t_f}{\tau_f}\right)$.
- For long times, $f(t_f) \sim t_f^{-(1+\alpha/2)}$ (famous $t_f^{-3/2}$ for $\alpha = 1$).

The first passage time

- $$sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) & x_0 < b, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > b. \end{cases}$$
- Solve, take the limit $p \to \infty$, and invert:
- $$\lim_{p\to\infty} G_0(p,t) = \Pr\{t_f > t\} = 1 - \int_0^t \frac{1}{\tau_f} l_{\alpha/2}\left(\frac{\tau}{\tau_f}\right) d\tau.$$
  - $\tau_f = (b^2/K_\alpha)^{1/\alpha}$.
  - $l_{\alpha/2}(t)$ is the one-sided Lévy distribution of order $\alpha/2$.
  - $\tilde{l}_{\alpha/2}(s) = e^{-s^{\alpha/2}}$.
- The first passage time PDF: $f(t_f) = \frac{1}{\tau_f} l_{\alpha/2}\left(\frac{t_f}{\tau_f}\right)$.
- For long times, $f(t_f) \sim t_f^{-(1+\alpha/2)}$ (famous $t_f^{-3/2}$ for $\alpha = 1$).

The first passage time

- $sG_{x_0}(p,s) - 1 = \begin{cases} K_\alpha s^{1-\alpha} \frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) & x_0 < b, \\ K_\alpha (s+p)^{1-\alpha} \frac{\partial^2}{\partial x_0^2} G_{x_0}(p,s) - pG_{x_0}(p,s) & x_0 > b. \end{cases}$
- Solve, take the limit $p \to \infty$, and invert:
- $\lim_{p\to\infty} G_0(p,t) = \Pr\{t_f > t\} = 1 - \int_0^t \frac{1}{\tau_f} l_{\alpha/2}\left(\frac{\tau}{\tau_f}\right) d\tau.$
  - $\tau_f = (b^2/K_\alpha)^{1/\alpha}$.
  - $l_{\alpha/2}(t)$ is the one-sided Lévy distribution of order $\alpha/2$.
  - $\tilde{l}_{\alpha/2}(s) = e^{-s^{\alpha/2}}$.
- The first passage time PDF: $f(t_f) = \frac{1}{\tau_f} l_{\alpha/2}\left(\frac{t_f}{\tau_f}\right)$.
- For long times, $f(t_f) \sim t_f^{-(1+\alpha/2)}$ (famous $t_f^{-3/2}$ for $\alpha = 1$).

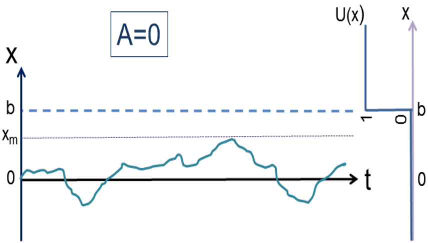
The maximal displacement

- Use exactly the same trick:

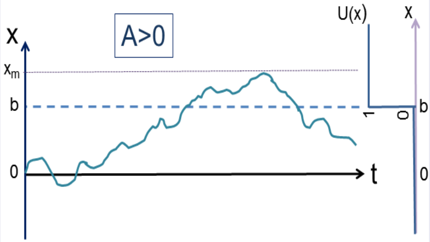
$$\Pr\left\{x_m \equiv \max_{0\le\tau\le t} x(\tau) < b\right\} = \lim_{p\to\infty} G_{x_0}(p,t).$$

- $P(x_m,t) = \frac{2}{\alpha\sqrt{K_\alpha}} \frac{t}{\left(x_m/\sqrt{K_\alpha}\right)^{1+2/\alpha}} l_{\alpha/2}\left[\frac{t}{\left(x_m/\sqrt{K_\alpha}\right)^{2/\alpha}}\right].$
- Exactly twice the PDF of x (for $x > 0$).

$x_m < b$

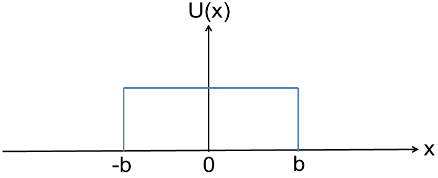


$x_m > b$



The time in $[-b, b]$

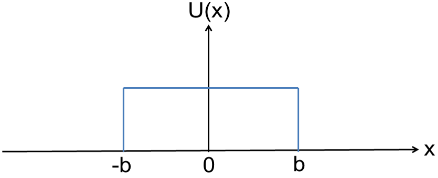
- $T_i \equiv \int_0^t U[x(\tau)]d\tau$.
- $U(x) = \begin{cases} 1 & -b < x < b, \\ 0 & \text{Otherwise.} \end{cases}$



- Solve backward equation in (p, s) space for $x_0 = 0$.
- $G_0(p,s) = \frac{p+s\left\{\cosh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]+\frac{(s+p)^{\alpha/2}}{s^{\alpha/2}}\sinh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]\right\}}{s(s+p)\left\{\cosh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]+\frac{(s+p)^{\alpha/2}}{s^{\alpha/2}}\sinh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]\right\}}$.
- $\langle T_i\rangle(s) = -\frac{\partial}{\partial p}G_0(p,s)\Big|_{p=0}$; $\langle T_i^2\rangle(s) = \frac{\partial^2}{\partial p^2}G_0(p,s)\Big|_{p=0}$.
- $\langle T_i\rangle \sim t^{1-\alpha/2}\frac{b}{\sqrt{K_\alpha}\Gamma(2-\alpha/2)}$.
- $\langle T_i^2\rangle \sim t^{2-\alpha/2}\frac{2b(1-\alpha)}{\sqrt{K_\alpha}\Gamma(3-\alpha/2)} + t^{2-\alpha}\frac{b^2(3\alpha-1)}{K_\alpha\Gamma(3-\alpha)}$.

The time in $[-b, b]$

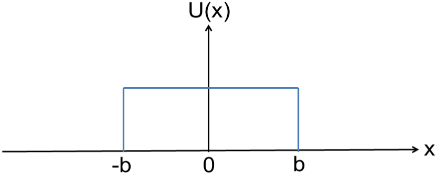
- $T_i \equiv \int_0^t U[x(\tau)]d\tau$.
- $U(x) = \begin{cases} 1 & -b < x < b, \\ 0 & \text{Otherwise.} \end{cases}$



- Solve backward equation in (p, s) space for $x_0 = 0$.
- $G_0(p,s) = \dfrac{p+s\left\{\cosh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]+\frac{(s+p)^{\alpha/2}}{s^{\alpha/2}}\sinh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]\right\}}{s(s+p)\left\{\cosh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]+\frac{(s+p)^{\alpha/2}}{s^{\alpha/2}}\sinh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]\right\}}$.
- $\langle T_i \rangle(s) = -\frac{\partial}{\partial p}G_0(p,s)\Big|_{p=0}$; $\langle T_i^2 \rangle(s) = \frac{\partial^2}{\partial p^2}G_0(p,s)\Big|_{p=0}$.
- $\langle T_i \rangle \sim t^{1-\alpha/2}\frac{b}{\sqrt{K_\alpha}\Gamma(2-\alpha/2)}$.
- $\langle T_i^2 \rangle \sim t^{2-\alpha/2}\frac{2b(1-\alpha)}{\sqrt{K_\alpha}\Gamma(3-\alpha/2)} + t^{2-\alpha}\frac{b^2(3\alpha-1)}{K_\alpha\Gamma(3-\alpha)}$.

The time in $[-b, b]$

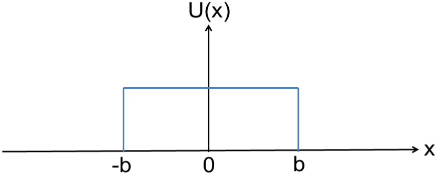
- $T_i \equiv \int_0^t U[x(\tau)]d\tau$.
- $U(x) = \begin{cases} 1 & -b < x < b, \\ 0 & \text{Otherwise.} \end{cases}$



- Solve backward equation in (p, s) space for $x_0 = 0$.
- $G_0(p,s) = \dfrac{p + s\left\{\cosh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right] + \frac{(s+p)^{\alpha/2}}{s^{\alpha/2}}\sinh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]\right\}}{s(s+p)\left\{\cosh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right] + \frac{(s+p)^{\alpha/2}}{s^{\alpha/2}}\sinh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]\right\}}$.
- $\langle T_i \rangle (s) = -\frac{\partial}{\partial p} G_0(p,s)\Big|_{p=0}$; $\langle T_i^2 \rangle (s) = \frac{\partial^2}{\partial p^2} G_0(p,s)\Big|_{p=0}$.
- $\langle T_i \rangle \sim t^{1-\alpha/2}\frac{b}{\sqrt{K_\alpha}\Gamma(2-\alpha/2)}$.
- $\langle T_i^2 \rangle \sim t^{2-\alpha/2}\frac{2b(1-\alpha)}{\sqrt{K_\alpha}\Gamma(3-\alpha/2)} + t^{2-\alpha}\frac{b^2(3\alpha-1)}{K_\alpha\Gamma(3-\alpha)}$.

The time in $[-b, b]$

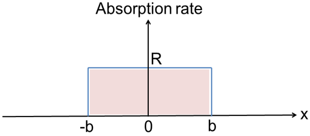
- $T_i \equiv \int_0^t U[x(\tau)]d\tau$.
- $U(x) = \begin{cases} 1 & -b < x < b, \\ 0 & \text{Otherwise.} \end{cases}$



- Solve backward equation in (p, s) space for $x_0 = 0$.
- $G_0(p,s) = \dfrac{p+s\left\{\cosh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]+\frac{(s+p)^{\alpha/2}}{s^{\alpha/2}}\sinh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]\right\}}{s(s+p)\left\{\cosh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]+\frac{(s+p)^{\alpha/2}}{s^{\alpha/2}}\sinh\left[\frac{(s+p)^{\alpha/2}}{\sqrt{K_\alpha}}b\right]\right\}}$.
- $\langle T_i \rangle (s) = -\frac{\partial}{\partial p} G_0(p,s)\Big|_{p=0}$; $\langle T_i^2 \rangle (s) = \frac{\partial^2}{\partial p^2} G_0(p,s)\Big|_{p=0}$.
- $\langle T_i \rangle \sim t^{1-\alpha/2} \frac{b}{\sqrt{K_\alpha}\Gamma(2-\alpha/2)}$.
- $\langle T_i^2 \rangle \sim t^{2-\alpha/2} \frac{2b(1-\alpha)}{\sqrt{K_\alpha}\Gamma(3-\alpha/2)} + t^{2-\alpha} \frac{b^2(3\alpha-1)}{K_\alpha \Gamma(3-\alpha)}$.

The average survival probability when the domain $[-b, b]$ is absorbing

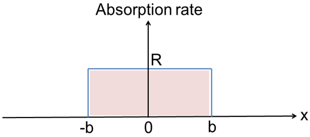
- The survival probability is $S = e^{-RT_i}$.
- $\langle S \rangle = \int_0^\infty e^{-RT_i} G_{x_0}(T_i, t) dT_i = G_{x_0}(R, t)$.



- We know $G_{x_0}(R, s)$ from the previous slide.
- We can invert for long times:
$\langle S \rangle \sim \left[\Gamma(1-\alpha/2)\sinh\left(\frac{bR^{\alpha/2}}{\sqrt{K_\alpha}}\right)\right]^{-1}(Rt)^{-\alpha/2} + \mathcal{O}\left[(Rt)^{-\alpha}\right]$.

The average survival probability when the domain $[-b, b]$ is absorbing

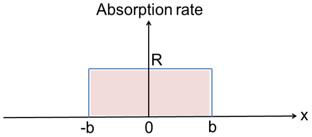
- The survival probability is $S = e^{-RT_i}$.
- $\langle S \rangle = \int_0^\infty e^{-RT_i} G_{x_0}(T_i, t) dT_i = G_{x_0}(R, t)$.



- We know $G_{x_0}(R, s)$ from the previous slide.
- We can invert for long times:
$\langle S \rangle \sim \left[\Gamma(1-\alpha/2) \sinh\left(\frac{bR^{\alpha/2}}{\sqrt{K_\alpha}}\right)\right]^{-1} (Rt)^{-\alpha/2} + \mathcal{O}\left[(Rt)^{-\alpha}\right]$.

The average survival probability when the domain $[-b, b]$ is absorbing

- The survival probability is $S = e^{-RT_i}$.
- $\langle S \rangle = \int_0^\infty e^{-RT_i} G_{x_0}(T_i, t) dT_i = G_{x_0}(R, t)$.



- We know $G_{x_0}(R, s)$ from the previous slide.
- We can invert for long times:
  $\langle S \rangle \sim \left[\Gamma(1-\alpha/2) \sinh\left(\frac{bR^{\alpha/2}}{\sqrt{K_\alpha}}\right)\right]^{-1} (Rt)^{-\alpha/2} + \mathcal{O}\left[(Rt)^{-\alpha}\right]$.

The functionals $U(x) = x^k$

- $sG(x,p,s) - \delta(x) = K_\alpha \frac{\partial^2}{\partial x^2}\left(s + px^k\right)^{1-\alpha} G(x,p,s) - px^k G(x,p,s)$.
- Operate on each term with $(-1)^n \frac{\partial^n}{\partial p^n}$, substitute $p = 0$, multiply each term by x^m, and integrate over all x.
- $\left\langle A_{x^k}^n x^m \right\rangle (s) = c_{n,m}(k) K_\alpha^{\frac{m+nk}{2}} s^{-\left(1+n+\frac{m+nk}{2}\alpha\right)}$.
- Substitute $m = 0$ and invert $s \to t$:
- $\left\langle A_{x^k}^n \right\rangle (t) = c_{n,0}(k) \frac{K_\alpha^{\frac{nk}{2}}}{\Gamma\left(1+n+\frac{n\alpha k}{2}\right)} t^{n\left(1+\frac{\alpha k}{2}\right)}$.
- Leads to the scaling form: $G(A_{x^k}, t) = \frac{1}{K_\alpha^{k/2} t^{1+\alpha k/2}} g_{\alpha,k}\left(\frac{A_{x^k}}{K_\alpha^{k/2} t^{1+\alpha k/2}}\right)$.

The functionals $U(x) = x^k$

- $sG(x,p,s) - \delta(x) = K_\alpha \frac{\partial^2}{\partial x^2}\left(s + px^k\right)^{1-\alpha} G(x,p,s) - px^k G(x,p,s)$.
- Operate on each term with $(-1)^n \frac{\partial^n}{\partial p^n}$, substitute $p = 0$, multiply each term by x^m, and integrate over all x.

- $\left\langle A_{x^k}^n x^m \right\rangle(s) =$
  $c_{n,m}(k) K_\alpha^{\frac{m+nk}{2}} s^{-\left(1+n+\frac{m+nk}{2}\alpha\right)}$.
- Substitute $m = 0$ and invert $s \to t$:
- $\left\langle A_{x^k}^n \right\rangle(t) =$
  $c_{n,0}(k) \frac{K_\alpha^{\frac{nk}{2}}}{\Gamma\left(1+n+\frac{n\alpha k}{2}\right)} t^{n\left(1+\frac{\alpha k}{2}\right)}$.
- Leads to the scaling form:
  $G(A_{x^k}, t) =$
  $\frac{1}{K_\alpha^{k/2} t^{1+\alpha k/2}} g_{\alpha,k}\left(\frac{A_{x^k}}{K_\alpha^{k/2} t^{1+\alpha k/2}}\right)$.

The functionals $U(x) = x^k$

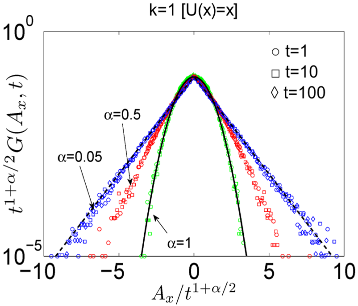
- $sG(x,p,s) - \delta(x) = K_\alpha \frac{\partial^2}{\partial x^2}\left(s + px^k\right)^{1-\alpha} G(x,p,s) - px^k G(x,p,s)$.
- Operate on each term with $(-1)^n \frac{\partial^n}{\partial p^n}$, substitute $p = 0$, multiply each term by x^m, and integrate over all x.

- $\left\langle A_{x^k}^n x^m \right\rangle(s) =$
  $c_{n,m}(k) K_\alpha^{\frac{m+nk}{2}} s^{-\left(1+n+\frac{m+nk}{2}\alpha\right)}$.
- Substitute $m = 0$ and invert $s \to t$:
- $\left\langle A_{x^k}^n \right\rangle(t) =$
  $c_{n,0}(k) \frac{K_\alpha^{\frac{nk}{2}}}{\Gamma\left(1+n+\frac{n\alpha k}{2}\right)} t^{n\left(1+\frac{\alpha k}{2}\right)}$.
- Leads to the scaling form:
  $G(A_{x^k}, t) =$
  $\frac{1}{K_\alpha^{k/2} t^{1+\alpha k/2}} g_{\alpha,k}\left(\frac{A_{x^k}}{K_\alpha^{k/2} t^{1+\alpha k/2}}\right)$.

The functionals $U(x) = x^k$

- $sG(x,p,s) - \delta(x) = K_\alpha \frac{\partial^2}{\partial x^2}\left(s + px^k\right)^{1-\alpha} G(x,p,s) - px^k G(x,p,s)$.
- Operate on each term with $(-1)^n \frac{\partial^n}{\partial p^n}$, substitute $p = 0$, multiply each term by x^m, and integrate over all x.

- $\left\langle A^n_{x^k} x^m \right\rangle (s) = c_{n,m}(k) K_\alpha^{\frac{m+nk}{2}} s^{-\left(1+n+\frac{m+nk}{2}\alpha\right)}$.
- Substitute $m = 0$ and invert $s \to t$:
- $\left\langle A^n_{x^k} \right\rangle (t) = c_{n,0}(k) \frac{K_\alpha^{\frac{nk}{2}}}{\Gamma\left(1+n+\frac{n\alpha k}{2}\right)} t^{n\left(1+\frac{\alpha k}{2}\right)}$.
- Leads to the scaling form: $G(A_{x^k}, t) = \frac{1}{K_\alpha^{k/2} t^{1+\alpha k/2}} g_{\alpha,k}\left(\frac{A_{x^k}}{K_\alpha^{k/2} t^{1+\alpha k/2}}\right)$.



Section 4

Functionals of bounded particles: weak ergodicity breaking

Ensemble and time averages

What is the average of $U(x)$ at time t in a closed system?

Ensemble average

$\langle U(x)\rangle_t = \int_{-\infty}^{\infty} G(x,t)U(x)dx.$

Time average

$\overline{U(x)}_t = \frac{1}{t}\int_0^t U[x(\tau)]d\tau.$

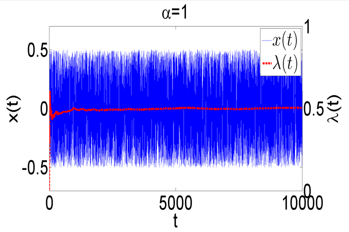
- In ergodic systems, for $t \to \infty$, $\langle U(x)\rangle = \overline{U(x)} = \int_{-\infty}^{\infty} G_{\rm eq}(x)U(x)dx.$
- When ergodicity is broken, $\overline{U(x)}$ is a random variable even for $t \to \infty$.

- Two examples...

Weak ergodicity breaking of the occupation time

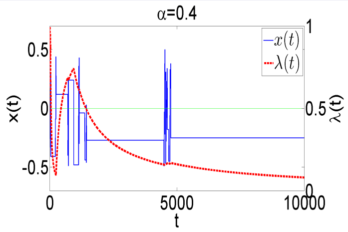
Consider the occupation fraction $\lambda = \frac{1}{t}\int_0^t \Theta[x(\tau)]d\tau$ for a particle in a box.

Normal diffusion



- Ergodic.
- For $t \to \infty$, $\lambda \to 1/2$.
- $\langle(\Delta\lambda)^2\rangle = 0$.

Anomalous diffusion

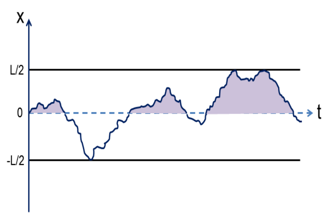


- Non-ergodic.
- Even for $t \to \infty$, λ is a random variable or $\langle(\Delta\lambda)^2\rangle > 0$.

The occupation time in half box

- The fluctuations of $\lambda = T_+/t$: $\left\langle (\Delta\lambda)^2 \right\rangle \sim \frac{1-\alpha}{4} + \frac{\alpha L^2}{24 K_\alpha \Gamma(3+\alpha)} t^{-\alpha}$.
- For $t \to \infty$ and $\alpha = 1$, $\left\langle (\Delta\lambda)^2 \right\rangle = 0$ and $G(\lambda) = \delta(\lambda - 1/2)$.
- For $t \to \infty$ and $\alpha < 1$, $\left\langle (\Delta\lambda)^2 \right\rangle = \frac{1-\alpha}{4} > 0$.
  $\Rightarrow$ weak ergodicity breaking!

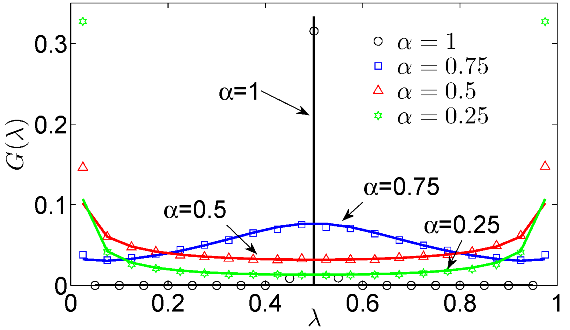
Illustration



- The PDF for long times is again Lamperti: $G_\alpha(\lambda) = \frac{\sin(\pi\alpha)}{\pi} \frac{\lambda^{\alpha-1}(1-\lambda)^{\alpha-1}}{\lambda^{2\alpha} + (1-\lambda)^{2\alpha} + 2\lambda^\alpha (1-\lambda)^\alpha \cos(\pi\alpha)}$.
- Here the exponent is α, compared to $\alpha/2$ for the free particle.

The occupation time in half box

Lamperti PDF

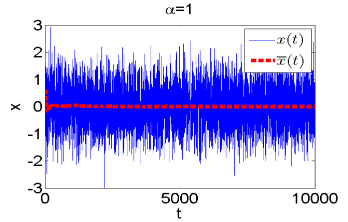


- For $\alpha = 1$, the particle uniformly covers the box in each trajectory.
- For small α, the particle remains in one side almost the entire time.

Weak ergodicity breaking of the time-averaged position

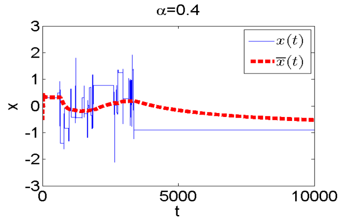
Consider the time-averaged position $\overline{x} = \frac{1}{t}\int_0^t x(\tau)d\tau$ in a harmonic field.

Normal diffusion



- Ergodic.
- For $t \to \infty$, $\overline{x} \to \langle x \rangle = 0$.
- $\left\langle (\Delta\overline{x})^2 \right\rangle = 0$.

Anomalous diffusion



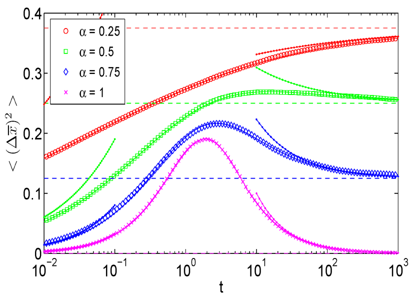
- Non-ergodic.
- Even for $t \to \infty$, $\overline{x}$ is a random variable or $\left\langle (\Delta\overline{x})^2 \right\rangle > 0$.

The fluctuations of the time-averaged position in a harmonic potential

- $\langle(\Delta\overline{x})^2\rangle = \langle x^2\rangle \{(1-\alpha) + 4\alpha E_{\alpha,3}[-(t/\tau)^\alpha] - 2(1+\alpha)E_{\alpha,3}[-2(t/\tau)^\alpha]\}$.
  - $E_{\alpha,3}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(3+\alpha n)}$ is the Mittag-Leffler function.
  - $\langle x^2\rangle \equiv K_\alpha \tau^\alpha = k_B T/(m\omega^2)$.

- For long times $\langle(\Delta\overline{x})^2\rangle \sim (1-\alpha)\langle x^2\rangle + \frac{(3\alpha-1)\langle x^2\rangle}{\Gamma(3-\alpha)}(t/\tau)^{-\alpha}$.
- For $t \to \infty$ and $\alpha = 1$, $\langle(\Delta\overline{x})^2\rangle = 0$ and $G(\overline{x}) = \delta(\overline{x})$.
- For $t \to \infty$ and $\alpha < 1$, $\langle(\Delta\overline{x})^2\rangle = (1-\alpha)\langle x^2\rangle > 0$.
  $\Rightarrow$ weak ergodicity breaking!

Simulations and theory



Section 5

Summary

Summary

- We developed a fractional Feynman-Kac equation for functionals of anomalous diffusion in the CTRW model.
- Using our equation, we could calculate several properties of anomalous diffusion such as the occupation time, the first passage time, the maximal displacement, and others.
- Our equation uncovers the kinetics of weak ergodicity breaking.

Summary

- We developed a fractional Feynman-Kac equation for functionals of anomalous diffusion in the CTRW model.
- Using our equation, we could calculate several properties of anomalous diffusion such as the occupation time, the first passage time, the maximal displacement, and others.
- Our equation uncovers the kinetics of weak ergodicity breaking.

Summary

- We developed a fractional Feynman-Kac equation for functionals of anomalous diffusion in the CTRW model.
- Using our equation, we could calculate several properties of anomalous diffusion such as the occupation time, the first passage time, the maximal displacement, and others.
- Our equation uncovers the kinetics of weak ergodicity breaking.

Thank you for your attention

Lior Turgeman, Shai Carmi, and Eli Barkai.
Fractional Feynman-Kac equation for non-Brownian functionals.
Phys. Rev. Lett. **103**, 190201 (2009).

Shai Carmi, Lior Turgeman, and Eli Barkai.
On distributions of functionals of anomalous diffusion paths.
J. Stat. Phys. **141**, 1071 (2010).

Shai Carmi and Eli Barkai.
A fractional Feynman-Kac equation for weak ergodicity breaking.
In preparation (2011).

Fractional Schrödinger equation

- We have seen that the Schrödinger equation in imaginary time is structurally similar to the Feynman-Kac equation.
- Is there a fractional Schrödinger equation?
- $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{-\hbar^2}{2m}\nabla_x^\mu\Psi(x,t) + V(x)\Psi(x,t)$.
- N. Laskin, *Phys. Rev. E* **62**, 3135 (2000).
- Analogous to Markovian CTRW with heavy tailed jumps.
- $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \mathcal{D}_{\mathrm{RL},t}^{1-\alpha}\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)\right]$.
- M. Naber, *J. Math. Phys.* **45**, 3339 (2004).
- Here $\mathcal{D}_{\mathrm{RL},t}^{1-\alpha}$ is the Riemann-Liouville operator ($s^{1-\alpha}$). Non-normalizable solution.
- Unclear if fractional Schrödinger equations have physical meaning.

Fractional Schrödinger equation

- We have seen that the Schrödinger equation in imaginary time is structurally similar to the Feynman-Kac equation.
- Is there a fractional Schrödinger equation?
- $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{-\hbar^2}{2m}\nabla_x^\mu\Psi(x,t) + V(x)\Psi(x,t)$.
- N. Laskin, *Phys. Rev. E* **62**, 3135 (2000).
- Analogous to Markovian CTRW with heavy tailed jumps.
- $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \mathcal{D}_{\mathrm{RL},t}^{1-\alpha}\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)\right]$.
- M. Naber, *J. Math. Phys.* **45**, 3339 (2004).
- Here $\mathcal{D}_{\mathrm{RL},t}^{1-\alpha}$ is the Riemann-Liouville operator ($s^{1-\alpha}$).
  Non-normalizable solution.
- Unclear if fractional Schrödinger equations have physical meaning.

Fractional Schrödinger equation

- We have seen that the Schrödinger equation in imaginary time is structurally similar to the Feynman-Kac equation.
- Is there a fractional Schrödinger equation?
- $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{-\hbar^2}{2m}\nabla_x^\mu\Psi(x,t) + V(x)\Psi(x,t)$.
- N. Laskin, *Phys. Rev. E* **62**, 3135 (2000).
- Analogous to Markovian CTRW with heavy tailed jumps.
- $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \mathcal{D}_{\mathrm{RL},t}^{1-\alpha}\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)\right]$.
- M. Naber, *J. Math. Phys.* **45**, 3339 (2004).
- Here $\mathcal{D}_{\mathrm{RL},t}^{1-\alpha}$ is the Riemann-Liouville operator ($s^{1-\alpha}$). Non-normalizable solution.
- Unclear if fractional Schrödinger equations have physical meaning.

Fractional Schrödinger equation

- We have seen that the Schrödinger equation in imaginary time is structurally similar to the Feynman-Kac equation.
- Is there a fractional Schrödinger equation?
- $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \frac{-\hbar^2}{2m}\nabla_x^\mu\Psi(x,t) + V(x)\Psi(x,t)$.
- N. Laskin, *Phys. Rev. E* **62**, 3135 (2000).
- Analogous to Markovian CTRW with heavy tailed jumps.
- $i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \mathcal{D}_{\mathrm{RL},t}^{1-\alpha}\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)\right]$.
- M. Naber, *J. Math. Phys.* **45**, 3339 (2004).
- Here $\mathcal{D}_{\mathrm{RL},t}^{1-\alpha}$ is the Riemann-Liouville operator ($s^{1-\alpha}$). Non-normalizable solution.
- Unclear if fractional Schrödinger equations have physical meaning.

Drift

- The particle experiences a constant force F.

Functional	**Result**
The first moment of the occupation time in $x > 0$	$\langle T_+ \rangle = t \left\{ 1 - \frac{1}{4\Gamma(2-\alpha)} (t/\tau)^{-\alpha} + \mathcal{O}\left[(t/\tau)^{-2\alpha} \right] \right\}$
The second moment of the occupation time in $x > 0$	$\left\langle {T_+}^2 \right\rangle = t^2 \left\{ 1 - \frac{1}{2\Gamma(3-\alpha)} (t/\tau)^{-\alpha} + \mathcal{O}\left[(t/\tau)^{-2\alpha} \right] \right\}$
The first moment of the time-averaged position	$\langle \overline{x} \rangle = \frac{\mu}{\Gamma(2+\alpha)} t^\alpha$
The second moment of the time-averaged position	$\left\langle \overline{x}^2 \right\rangle = \frac{4K_\alpha}{\Gamma(3+\alpha)} t^\alpha + \frac{2\mu^2(2+\alpha)}{\Gamma(3+2\alpha)} t^{2\alpha}$

- $\tau^\alpha = \frac{4(k_B T)^2}{K_\alpha F^2}$; $\mu = \frac{K_\alpha F}{k_B T}$.

The long times PDF $G_{\text{eq}}(\overline{U})$

- Consider a functional $A = \int_0^t U[x(\tau)]d\tau$, and $\overline{U} = A/t$.
- For long times, we can show that
  $K_\alpha \left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F(x)}{k_BT}\right][s+pU(x)]^{1-\alpha}G(x,p,s) = 0.$
- Integrate the forward equation over all x to obtain:
  $\int_{-\infty}^{\infty}[s+pU(x)]\,G(x,p,s)dx = 1.$
- This gives: $G(x,p,s) = \frac{[s+pU(x)]^{\alpha-1}e^{-V(x)/(k_BT)}}{\int_{-\infty}^{\infty}[s+pU(x)]^{\alpha}e^{-V(x)/(k_BT)}dx}.$
- Invert: $G_{\text{eq}}(\overline{U}) = \frac{\sin(\pi\alpha)}{\pi}\frac{I_{\alpha-1}^{<}(\overline{U})I_{\alpha}^{>}(\overline{U})+I_{\alpha-1}^{>}(\overline{U})I_{\alpha}^{<}(\overline{U})}{[I_{\alpha}^{>}(\overline{U})]^2+[I_{\alpha}^{<}(\overline{U})]^2+2\cos(\pi\alpha)I_{\alpha}^{>}(\overline{U})I_{\alpha}^{<}(\overline{U})},$
- where
  $I_\alpha^{<}(\overline{U}) = \int_{\overline{U}<U(x)} e^{-V(x)/(k_BT)}\left[U(x)-\overline{U}\right]^{\alpha}dx,$
  $I_\alpha^{>}(\overline{U}) = \int_{\overline{U}>U(x)} e^{-V(x)/(k_BT)}\left[\overline{U}-U(x)\right]^{\alpha}dx.$

The long times PDF $G_{\text{eq}}(\overline{U})$

- Consider a functional $A = \int_0^t U[x(\tau)]d\tau$, and $\overline{U} = A/t$.
- For long times, we can show that
$K_\alpha \left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F(x)}{k_B T}\right][s + pU(x)]^{1-\alpha} G(x,p,s) = 0.$
- Integrate the forward equation over all x to obtain:
$\int_{-\infty}^{\infty} [s + pU(x)]\, G(x,p,s)dx = 1.$
- This gives: $G(x,p,s) = \frac{[s+pU(x)]^{\alpha-1}e^{-V(x)/(k_B T)}}{\int_{-\infty}^{\infty}[s+pU(x)]^{\alpha}e^{-V(x)/(k_B T)}dx}$.
- Invert: $G_{\text{eq}}(\overline{U}) = \frac{\sin(\pi\alpha)}{\pi}\frac{I_{\alpha-1}^{<}(\overline{U})I_{\alpha}^{>}(\overline{U}) + I_{\alpha-1}^{>}(\overline{U})I_{\alpha}^{<}(\overline{U})}{[I_{\alpha}^{>}(\overline{U})]^2 + [I_{\alpha}^{<}(\overline{U})]^2 + 2\cos(\pi\alpha)I_{\alpha}^{>}(\overline{U})I_{\alpha}^{<}(\overline{U})}$,
- where
$I_{\alpha}^{<}(\overline{U}) = \int_{\overline{U}<U(x)} e^{-V(x)/(k_B T)}\left[U(x) - \overline{U}\right]^{\alpha} dx,$
$I_{\alpha}^{>}(\overline{U}) = \int_{\overline{U}>U(x)} e^{-V(x)/(k_B T)}\left[\overline{U} - U(x)\right]^{\alpha} dx.$

The long times PDF $G_{\text{eq}}(\overline{U})$

- Consider a functional $A = \int_0^t U[x(\tau)]d\tau$, and $\overline{U} = A/t$.
- For long times, we can show that
$K_\alpha \left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F(x)}{k_B T}\right] [s + pU(x)]^{1-\alpha} G(x,p,s) = 0.$
- Integrate the forward equation over all x to obtain:
$\int_{-\infty}^{\infty} [s + pU(x)]\, G(x,p,s)dx = 1.$
- This gives: $G(x,p,s) = \frac{[s+pU(x)]^{\alpha-1} e^{-V(x)/(k_B T)}}{\int_{-\infty}^{\infty} [s+pU(x)]^{\alpha} e^{-V(x)/(k_B T)} dx}$.
- Invert: $G_{\text{eq}}(\overline{U}) = \frac{\sin(\pi\alpha)}{\pi} \frac{I_{\alpha-1}^{<}(\overline{U}) I_\alpha^{>}(\overline{U}) + I_{\alpha-1}^{>}(\overline{U}) I_\alpha^{<}(\overline{U})}{[I_\alpha^{>}(\overline{U})]^2 + [I_\alpha^{<}(\overline{U})]^2 + 2\cos(\pi\alpha) I_\alpha^{>}(\overline{U}) I_\alpha^{<}(\overline{U})}$,
- where
$I_\alpha^{<}(\overline{U}) = \int_{\overline{U}<U(x)} e^{-V(x)/(k_B T)} \left[U(x) - \overline{U}\right]^\alpha dx,$
$I_\alpha^{>}(\overline{U}) = \int_{\overline{U}>U(x)} e^{-V(x)/(k_B T)} \left[\overline{U} - U(x)\right]^\alpha dx.$

The long times PDF $G_{\text{eq}}(\overline{U})$

- Consider a functional $A = \int_0^t U[x(\tau)]d\tau$, and $\overline{U} = A/t$.
- For long times, we can show that
$K_\alpha \left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F(x)}{k_B T}\right] [s + pU(x)]^{1-\alpha} G(x,p,s) = 0.$
- Integrate the forward equation over all x to obtain:
$\int_{-\infty}^{\infty} [s + pU(x)]\, G(x,p,s)dx = 1.$
- This gives: $G(x,p,s) = \frac{[s+pU(x)]^{\alpha-1} e^{-V(x)/(k_B T)}}{\int_{-\infty}^{\infty}[s+pU(x)]^{\alpha} e^{-V(x)/(k_B T)}dx}$.
- Invert: $G_{\text{eq}}(\overline{U}) = \frac{\sin(\pi\alpha)}{\pi} \frac{I_{\alpha-1}^{<}(\overline{U}) I_{\alpha}^{>}(\overline{U}) + I_{\alpha-1}^{>}(\overline{U}) I_{\alpha}^{<}(\overline{U})}{[I_{\alpha}^{>}(\overline{U})]^2 + [I_{\alpha}^{<}(\overline{U})]^2 + 2\cos(\pi\alpha) I_{\alpha}^{>}(\overline{U}) I_{\alpha}^{<}(\overline{U})}$,
- where
$I_{\alpha}^{<}(\overline{U}) = \int_{\overline{U}<U(x)} e^{-V(x)/(k_B T)} \left[U(x) - \overline{U}\right]^{\alpha} dx,$
$I_{\alpha}^{>}(\overline{U}) = \int_{\overline{U}>U(x)} e^{-V(x)/(k_B T)} \left[\overline{U} - U(x)\right]^{\alpha} dx.$

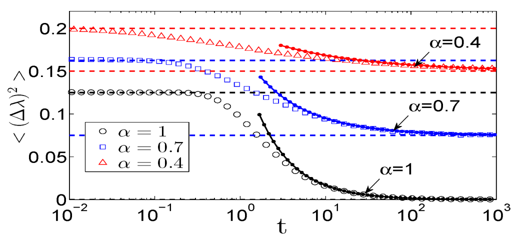
The long times PDF $G_{\mathrm{eq}}(\overline{U})$

- Consider a functional $A = \int_0^t U[x(\tau)]d\tau$, and $\overline{U} = A/t$.
- For long times, we can show that $K_\alpha \left[\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\frac{F(x)}{k_B T}\right][s + pU(x)]^{1-\alpha}G(x,p,s) = 0$.
- Integrate the forward equation over all x to obtain: $\int_{-\infty}^{\infty}[s + pU(x)]\,G(x,p,s)dx = 1$.
- This gives: $G(x,p,s) = \frac{[s+pU(x)]^{\alpha-1}e^{-V(x)/(k_B T)}}{\int_{-\infty}^{\infty}[s+pU(x)]^{\alpha}e^{-V(x)/(k_B T)}dx}$.
- Invert: $G_{\mathrm{eq}}(\overline{U}) = \frac{\sin(\pi\alpha)}{\pi}\frac{I_{\alpha-1}^{<}(\overline{U})I_{\alpha}^{>}(\overline{U})+I_{\alpha-1}^{>}(\overline{U})I_{\alpha}^{<}(\overline{U})}{[I_{\alpha}^{>}(\overline{U})]^2+[I_{\alpha}^{<}(\overline{U})]^2+2\cos(\pi\alpha)I_{\alpha}^{>}(\overline{U})I_{\alpha}^{<}(\overline{U})}$,
- where $I_{\alpha}^{<}(\overline{U}) = \int_{\overline{U}<U(x)} e^{-V(x)/(k_B T)}\left[U(x) - \overline{U}\right]^{\alpha}dx$,
  $I_{\alpha}^{>}(\overline{U}) = \int_{\overline{U}>U(x)} e^{-V(x)/(k_B T)}\left[\overline{U} - U(x)\right]^{\alpha}dx$.

The dynamics of the occupation time in a box

- For long times, the PDF is Lamperti with index α.
- $\langle(\Delta\lambda)^2\rangle \to \frac{\alpha L^2}{24K_\alpha\Gamma(3+\alpha)}t^{-\alpha} + \frac{1-\alpha}{4}$.
- For short times, the PDF is Lamperti with index $\alpha/2$ (free particle).
- $\langle(\Delta\lambda)^2\rangle \to \frac{1-\alpha/2}{4}$.

Simulations and theory



The first passage time in a box

- The time t_f when a particle starting at $x_0 = -b$ $(0 < b < L/2)$ first hits $x = 0$ is the first passage time in a box.
- Use Kac's trick again:
- $\Pr\{t_f > t\} = \Pr\left\{\max_{0\le\tau\le t} x(\tau) < 0\right\} = \lim_{p\to\infty} G_{x_0}(p,t)$
- Here, $U(x) = \Theta(x)$.
- The solution of the backward equation gives
  $\lim_{p\to\infty} G_{-b}(p,s) = \frac{1}{s}\left\{1 - \frac{\cosh\left[\left(\frac{L}{2}-b\right)\frac{s^{\alpha/2}}{\sqrt{K_\alpha}}\right]}{\cosh\left(\frac{Ls^{\alpha/2}}{2\sqrt{K_\alpha}}\right)}\right\}.$
- The first passage time PDF is $f(t_f) = \frac{b(L-b)}{2K_\alpha|\Gamma(-\alpha)|} t_f^{-(1+\alpha)}$.
- Here, $f(t_f) \sim t_f^{-(1+\alpha)}$ compared to $f(t_f) \sim t_f^{-(1+\alpha/2)}$ for a free particle.
- For $\alpha < 1$, $\langle t_f \rangle = \infty$.

The PDF of the time-averaged position

- We know the PDF for long times (long expression).
- For short times the PDF is equivalent to that of free particles.
- For $0 < \alpha < 1/3$ the PDF becomes wider in time.
- For $1/3 < \alpha < 1$ the PDF is first narrow (initial condition), then wide, then narrow again (but with finite width for $\alpha < 1$).

Simulations and theory